# Parallel Inverse Aggregate Demand Curves in Discrete Choice Models

Kory Kroft, René Leal-Vizcaíno, Matthew J. Notowidigdo, and Ting Wang*

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**Abstract**

This paper shows that commonly-used discrete choice models feature parallel market demands. In particular, in random utility models, a necessary and sufficient condition for inverse aggregate demand curves to shift in parallel with respect to variety is that the random utility shocks follow the Gumbel distribution. Using results from Extreme Value Theory, we provide a set of conditions under which assuming "parallel inverse demand curves" may be a good approximation to the actual shift in aggregate demand from a change in variety. We provide simulations illustrating these theoretical results, and we discuss an application that uses the parallel demands assumption to estimate the change in consumer surplus from an exogenous change in variety.

*Kroft: University of Toronto and NBER, kory.kroft@utoronto.ca; Leal-Vizcaíno: Northwestern University and Bank of Mexico, renelealv@u.northwestern.edu; Notowidigdo: Northwestern University and NBER, noto@northwestern.edu; Wang: Northwestern University, tingwang@u.northwestern.edu. We thank Simon Anderson, Alex Frankel, and Rob Porter for helpful comments. We gratefully acknowledge funding from the Social Sciences and Humanities Research Council (SSHRC). Any opinion, findings, and conclusions or recommendations expressed in this material are those of the authors(s) and do not necessarily reflect the views of the SSHRC.

# 1 Introduction

This paper shows that inverse market demand curves satisfy a parallel inverse aggregate demand property in commonly-used discrete choice models – hereafter referred to as "parallel demands". This property implies that inverse aggregate demand curves shift vertically in parallel in response to an exogenous change in the number of varieties in a market. This paper makes two theoretical contributions. First, we show that in random utility models with i.i.d. random utility shocks, a necessary and sufficient condition for parallel demands is that the random utility shocks are distributed according to the Gumbel distribution.

Our second theoretical contribution is to demonstrate that the parallel demands assumption is valid for a wide class of discrete choice models. In particular, we show that for a broad set of distributions of the random utility shock, the inverse aggregate demands are asymptotically parallel as the number of varieties increases. This result comes from Extreme Value Theory (EVT): when the random utility shocks are independent and identically distributed, the distribution of the maximum order statistic converges to a Gumbel distribution for a wide range of distributions. We show numerically that this convergence happens very quickly for many standard distributions (e.g., Normal, Gamma, and Exponential).[1]

We also show that our theoretical results remain valid in models that allow different substitutability within the market with product variety (relative to the outside option), as well as correlated tastes across products within the variety market. Such models capture more realistic substitution patterns and have been studied and used in many literatures (e.g., McFadden, 1978; Cardell, 1997; Berry, 1994). We illustrate this results for the particular case of the Nested Logit model (Cardell, 1997).

Lastly, we show how the parallel demands property is useful when evaluating the change in consumer surplus from an exogenous change in the number of varieties, which we label as the "variety effect". Under parallel demands, the change in consumer surplus can be identified

[1] Extreme Value Theory has been used in economics in a random utility context by Gabaix et. al. (2016) to show that there might exist robustly high equilibrium markups in large markets that are insensitive to the degree of competition as the number of firms increases.

using two sources of variation that are orthogonal to preferences and other determinants of demand, one source that only affects prices in the market and another source that affects both variety and prices (see Kroft et al. 2019 for more details).

Beyond the study of consumer choice, discrete choice models are widespread in economics, and our theoretical results may therefore be usefully applied to other economic settings, such as the choice of neighborhood (McFadden 1978; Bayer, Ferreira and McMillan 2007), occupation (Hsieh et al 2013), firm (Card et al 2018; Chan, Kroft and Mourifie 2019; Lamadon, Mogstad and Setzler 2019), and school (Dinerstein and Smith 2014). In all of these settings, the welfare effects corresponding to changes in the number of choices (or "varieties") may be calculated using the approach described in this paper as long as the parallel demands assumption holds. For example, the parallel demands assumption allows for the willingness-to-pay for variety (estimated among the set of marginal consumers) to inform the average willingness-to-pay for variety (across all of the infra-marginal consumers). As a result, the parallel demands assumption may simplify welfare analysis by facilitating extrapolation to infra-marginal individuals without needing to specify a full structural model.[2]

The rest of the paper proceeds as follows. Section 2 derives the main theoretical results, including the the necessary and sufficient condition for parallel demands to exist and the validity of the asymptotic approximation. Section 3 considers a generalization to the Nested Logit model and discusses an application of parallel demands to estimating consumers' "love of variety". Section 4 concludes.

# 2 Parallel Demands

In this section, we derive conditions under which inverse market demands are exactly parallel and also conditions where assuming parallel demands is likely to be a good approximation. We first define parallel demands and state our first theorem, and then we characterize a class of models that satisfy the asymptotic approximation of parallel demands.

[2] Of course, such straightforward extrapolation using parallel demands does not come without costs; for example, the conditions that are needed for parallel demands may not hold in all empirical settings.

## 2.1 Necessary and Sufficient Conditions

Consider a unit mass population of ex ante identical and independent consumers indexed by $i$. Consumers either choose to purchase a single product $j \in \{1, ..., J\}$ – where $J$ is defined as the number of product varieties – or choose the outside option $j = 0$.

*Preferences.* The indirect utility of individual $i$ who purchases product $j$ is given by:

$$u_{ij}(y_i, p_j) = \alpha(y_i - p_j) + \delta_j + \varepsilon_{ij} \tag{1}$$

where scalar $\alpha$ is the marginal utility of income, $y_i$ is the consumer's income, $p_j$ is the price of good $j$, $\delta_j$ is the product quality, and $\varepsilon_{ij}$ is an idiosyncratic match value between consumer $i$ and product $j$, which captures heterogeneity in tastes across consumers and products. The utility of individual $i$ who chooses the outside option is given by $u_{i0} = \alpha y_i + \varepsilon_{i0}$.

*Demand.* Given the indirect utility function in equation (1), assume there is 0 probability of ties. For each consumer, we define the demand for product $j$ as $q_j(p_1, \ldots, p_J, J) : \mathbb{R}_+^{J+1} \to \mathbb{R}_+$, and express it as

$$q_j(p_1, \ldots, p_J, J) = \mathbb{P}\left(u_{ij}(y_i, p_j) = \max_{j' \in \{0,\ldots,J\}} u_{ij'}(y_i, p_{j'})\right). \tag{2}$$

*Aggregate Demand.* We may express aggregate demand for all products excluding the outside good when $J$ varieties are available as $Q(p_1, \ldots, p_J, J) : \mathbb{R}_+^{J+1} \to \mathbb{R}_+$ , which takes the form

$$Q(p_1, \ldots, p_J, J) = \sum_{j=1}^{J} q_j(p_1, \ldots, p_J, J). \tag{3}$$

Specifying a distribution for the random utility shocks ($\varepsilon_{ij}$) from the Generalized Extreme Value (GEV) family gives rise to different models of discrete choice.[3] To better illustrate our results and to simplify the derivation, we impose the following symmetry assumptions:

**Assumption 1.** *We assume that (1) the random utility shocks $(\varepsilon_{ij}), j = 1...J$ are continuously, independently, and identically distributed (i.i.d.), and are independent of the distribution of $\epsilon_{i0}$, $y_i$, and $\delta_j, j = 1...J$; (2) products are symmetric, meaning $\delta_j = \delta$.*

The above assumptions imply that product prices will be identical in equilibrium ($p_j = p$)

[3]See Train (2003) for more details.

under the additional assumption of identical production costs. With symmetry in prices, we can re-write the demand function $q(p, J) : \mathbb{R}_+^2 \to \mathbb{R}_+$ and the aggregate demand function $Q(p, J) : \mathbb{R}_+^2 \to \mathbb{R}_+$ as

$$q(p, J) = \mathbb{P}\left(u_{ij}(y_i, p) = \max_{j' \in \{0,\dots,J\}} u_{ij'}(y_i, p)\right)$$

$$Q(p, J) = Jq(p, J)$$

Next, we can solve for the inverse aggregate demand function $P(Q, J) : \mathbb{R}_+^2 \to \mathbb{R}_+$ by inverting $Q(p, J)$ with respect to $p$.[4] We can now introduce our definition of the parallel inverse aggregate demand property:

**Definition 1.** A discrete choice model is said to give rise to parallel demands if for all $J$, $J' \neq J$, and $Q$

$$\frac{\partial P}{\partial Q}(Q, J) = \frac{\partial P}{\partial Q}(Q, J'),$$

where $P(Q, J)$ is the inverse aggregate demand function, and $J$ and $J'$ are any numbers of product varieties.

In other words, the inverse aggregate demand curve shifts vertically "in parallel" if the number of varieties $J$ and the aggregate demand $Q$ are separable in the inverse aggregate demand expression. If we interpret $P(Q, J)$ as the Willingness-To-Pay (WTP), then parallel demands imply that the change in WTP of the marginal consumer is equal to the change in WTP of the average consumer in the market. Using this definition, we can now state the following theorem:

**Theorem 1.** *Let the random utility shocks $(\varepsilon_{ij})_{j=1}^{\infty}$ be i.i.d. and also independent of the distribution of $\varepsilon_{i0}$. Then, for any distribution of $\varepsilon_{i0}$, a necessary and sufficient condition for parallel demands is that the random utility shocks $(\varepsilon_{ij})_{j=1}^{\infty}$ follow the Gumbel distribution.*

**Proof**: We prove the sufficiency in the remainder of this subsection, and we prove necessity in the Appendix. For sufficiency, we use a standard Logit model and show that it features parallel demands. In equation (1), if the random utility shocks $(\varepsilon_{ij})$ are drawn from the

[4] $Q(p, J)$ is invertible, because it is a strictly decreasing function with respect to $p$.

Gumbel distribution, then this model corresponds to a multinomial Logit model in which there are $J+1$ products. Then, for any $j \in \{1, ..., J\}$

$$q(p, J) = \frac{e^{\delta - \alpha p}}{1 + J e^{\delta - \alpha p}}.$$

The aggregate demand of the variety market is equal to

$$Q(p, J) = \frac{J e^{\delta - \alpha p}}{1 + J e^{\delta - \alpha p}}.$$

Inverting the above expression, we find that the inverse aggregate demand curve of the Logit model is given by

$$P(Q, J) = \frac{\delta}{\alpha} + \frac{1}{\alpha} \log J - \frac{1}{\alpha} \log \left( \frac{Q}{1 - Q} \right)$$

which is separable in $J$ and $Q$. This implies that exogenous shifts in variety shift the inverse aggregate demand curve in parallel.

Consumers' "love of variety" is characterized by $\frac{dP}{dJ} = \frac{1}{\alpha J}$. When $\frac{1}{\alpha J}$ is large, consumers have a high value of variety. If we define the market share $s = \frac{1}{J}$, we can re-write "love of variety" as $\frac{1}{\alpha} s$ . As the variety gets large ($\frac{1}{J} \to 0$), the market share of each product becomes very small and the value of additional products is essentially 0. Note that the love of variety parameter $\frac{1}{\alpha}$ is inversely related to the elasticity of substitution considered in trade models featuring a representative consumer with a CES utility function. In fact, the Logit model aggregates to the CES model if we substitute $log(p)$ instead of prices $p$ into the indirect utility function above (see Anderson, de Palma and Thisse 1987).[5]

We have thus shown that when the random utility shocks ($\varepsilon_{ij}$) are i.i.d. Gumbel and independent of $\varepsilon_{i0}$ (which does not necessarily need to follow the Gumbel distribution), then we obtain parallel demands. In fact, Theorem 1 states that if random utility shocks are i.i.d., then it is also a necessary condition that the random utility shocks ($\varepsilon_{ij}$) are distributed Gumbel in order to get parallel demands. The proof in the Appendix uses the Extreme Value Theorem to prove this result.

[5]The formal connection requires one to introduce a second stage where individuals choose a continuous quantity of the good.

## 2.2 Asymptotic Approximations

The previous section showed that Gumbel random utility shocks is both necessary and sufficient for parallel demands. Using Extreme Value Theory, we now show that there is a large class of models beyond Gumbel that admit parallel demands asymptotically (as $J$ grows large). The random utility models in this class have in common that the distribution of the maxima of the shocks is asymptotically Gumbel, which implies that the aggregate inverse demands are asymptotically parallel. We now define a class of models that admit this asymptotic approximation, and we provide a sufficient condition to show that a given random utility model is in this class.

**Definition 2.** Let $(\varepsilon_j)$ be i.i.d. distributed according to a continuous cdf $F$. We say that $F$ is in the domain of attraction of the Gumbel distribution if

$$\max_{j\in\{1,\ldots,J\}} \varepsilon_j \overset{a}{\sim} Gumbel(\mu(J), \eta(J)),$$

as $J \to \infty$ for some location and dispersion parameters $(\mu(J), \eta(J))$ .

**Lemma 1.** *Let $x_0$ be the supremum of the support of a cdf $F$ that is twice continuously differentiable. If $F$ satisfies that $\lim_{x\to x_0} \frac{F''(x)(1-F(x))}{F'^2} = -1$ then $F$ is in the domain of attraction of the Gumbel distribution.*

See Resnick (1987) for a proof of the lemma and a full characterization of the domain of attraction of the Gumbel distribution. The characterization is outside the scope of the paper, and the lemma is enough for our purposes. For example, if $(\varepsilon_j)$ are i.i.d. $N(0, \eta^2)$ or exponential, then the above lemma applies. The next theorem provides the microfoundation for our key assumption of parallel demands, and states that inverse demands become parallel as variety increases for any random utility model with shocks in the Gumbel domain of attraction.

**Theorem 2.** *Let the random utility shocks $(\varepsilon_j)_{j=1}^{\infty}$ be i.i.d. and distributed according to $F$ in the domain of attraction of the Gumbel distribution. Then, for any large enough $J$ and $K$ there exists $d$ such that for all $p \in \mathbb{R}$ we have $Q(p, J) \approx Q(p + d, K)$. Specifically, for all $p$*

*we have*

$$Q(p,J) = \mathbb{P}\left(\max_{j\in\{1,\ldots,J\}} u_{ij}(p) > u_{i0}\right) \overset{a}{\sim} \mathbb{P}\left(\max_{j\in\{1,\ldots,K\}} u_{ij}(p+d) > u_{i0}\right) = Q(p+d,K)$$

*Therefore the inverse demands are approximately parallel* $P(Q,K) \approx P(Q,J) + d$ *for all* $Q$*, for large enough* $J$ *and* $K$*.*

**Proof**: See Appendix.

In Section 3.3, we numerically simulate different random utility models, which allows us to assess the approximation theorem by computing the "bias" from assuming parallel demands.

# 3 Generalization and Applications

In this section we revise the model in 2.1 and generalize it to a Nested Logit model. While preserving the extreme value distribution of consumers' tastes within the inside market, the Nested Logit model in many cases better captures the substitution patterns of products by allowing different substitutability within the variety market relative to the outside option and correlated tastes across products within the variety market. We show that in this model, demands are parallel since the distribution of random utility shocks satisfies the necessary and sufficient condition.

We then consider an application of parallel demands that is developed in detail in Kroft et al. (2019). With parallel demands, we show that the welfare effect from changes in varieties can be measured by a sufficient statistics approach. There are several advantages of this approach. First, since the method is based on the aggregate demand, one does not need firm-level or product-level prices and expenditure shares to calculate welfare effects. Second, as Theorem 2 suggests, this method is applicable to a wide class of discrete choice models. One does not need to specify the underlying preferences as long as the taste shocks are assumed to be distributed according to the theorem.

## 3.1 Nested Logit Model

Similar to the multinomial Logit model, we consider a population of statistically identical and independent consumers indexed by $i$ of mass unity who choose to purchase a single product $j \in \{1, ..., J\}$ or the outside option $j = 0$.

*Preferences.* The indirect utility of individual $i$ who purchases product $j$ is now given by:

$$u_{ij}(y_i, p_j) = \alpha(y_i - p_j) + \delta_j + (1-\sigma)\nu_i + \sigma\varepsilon_{ij} \tag{4}$$

where $(1-\sigma)\nu_i + \sigma\varepsilon_{ij}$ is the idiosyncratic match value between consumer $i$ and product $j$, which captures heterogeneity in tastes across consumers and products, and correlation in tastes across products. The utility of individual $i$ who chooses the outside option is still given by $u_{i0} = \alpha y_i + \varepsilon_{i0}$. Similar to the Logit model, we make the following assumptions.

**Assumption 2.** *We assume that (1) for $j \neq 0$, the random utility shocks $(\varepsilon_{ij}), j = 1...J$ are continuously, independently and identically distributed (i.i.d.) and independent of $\varepsilon_{i0}$, $y_i$, $\nu_i$, and $\delta_j, j = 1...J$, but we allow $\varepsilon_{i0}$ to be correlated with $\nu_i$; (2) products are symmetric $\delta_j = \delta$.*

Demand, aggregate demand, and inverse demand have the same expressions as in Section 2.1. Given symmetry assumptions we can express the aggregate demand as

$$Q(p, J) = 1 - \mathbb{P}\left(\max_{j\in\{1,...,J\}} \varepsilon_j \leq \alpha p - \frac{\delta}{\sigma} - \frac{1-\sigma}{\sigma}\nu + \frac{1}{\sigma}\varepsilon_0\right).$$

The parameter $\sigma$ captures the correlated tastes across products of the inside market. In addition to conditions in Theorem 1, by assuming the independence of $\epsilon_j$ from $\nu$ and $\sigma$, we are able to provide a corollary to accommodate such correlated preferences.

**Corollary 1.** *Let the random utility shocks $(\varepsilon_{ij})$ be i.i.d., independent of the size of $\sigma$, the distribution of $\nu_i$ and the distribution of $\varepsilon_{i0}$. Then, a necessary and sufficient condition for parallel demands is that the random utility shocks $(\varepsilon_{ij})$ follow Gumbel distribution.*

**Proof**: See Appendix.

We use the Nested Logit model to illustrate this result. In equation (4), if the random utility shocks $(\varepsilon_{ij})$ are drawn from the Gumbel distribution, and $(1-\sigma)\nu_i$ has the distribution derived in Cardell (1997)[6], then this model corresponds to the Nested Logit model in which there are only two nests: one which includes $j = 1, ..., J$ and the other which includes only the outside option $j = 0$. Then

$$q_i(p_1, \ldots, p_J, J) = \frac{\left(\sum_{j=1}^{J} e^{\frac{\delta_j - \alpha p_j}{\sigma}}\right)^{\sigma}}{1 + \left(\sum_{j=1}^{J} e^{\frac{\delta_j - \alpha p_j}{\sigma}}\right)^{\sigma}} \frac{e^{\frac{\delta_i - \alpha p_i}{\sigma}}}{\sum_{j=1}^{J} e^{\frac{\delta_j - \alpha p_j}{\sigma}}}.$$

As the parameter $\sigma$ goes to 0, the only random term in equation (4) is $\nu_i$ which is constant across all $j \neq 0$. When $\sigma = 1$, we retrieve the Logit model. The parameter $\sigma/\alpha$ characterizes consumers' "love of variety". With the symmetry assumption, aggregate demand is equal to:

$$Q(p, J) = \frac{J^{\sigma} e^{\delta - \alpha p}}{1 + J^{\sigma} e^{\delta - \alpha p}}.$$

Inverting the above expression, we find that the inverse aggregate demand curve is given by

$$P(Q, J) = \frac{\delta}{\alpha} + \frac{\sigma}{\alpha} \log J - \frac{1}{\alpha} \log\left(\frac{Q}{1 - Q}\right).$$

This implies that exogenous shifts in variety move the inverse aggregate demand curve in parallel. As with the Logit model, we now can measure the "love of variety" by $\frac{dP}{dJ} = \frac{\sigma}{\alpha J} = \frac{\sigma}{\alpha} s$, where $s$ is again the market share of each product. As $\sigma$ goes to 0, taste shocks of the inside market are more strongly correlated; i.e., products are more substitutable, so the value of additional products becomes less.

As a corollary to this section, one might suggest that the outside option should always be considered as part of a separate nest in discrete choice models, so as to let not only variation in prices but variation in the number of products $J$ to identify the "love of variety".

## 3.2 Variety Effect

This section describes an application of parallel demands to study the welfare gains of consumers from the introduction of new product. Estimating the value of new products is important for a broad range of economic issues, ranging from a full accounting of the gains from

[6]See Cardell (1997) for the class of $C(\cdot)$ distributions. This specified class of distributions makes the combined idiosyncratic shocks distributed Type I extreme, and thus allows us to write the demand in a closed form. It is not necessary for parallel demands, as the proof of Theorem 3 suggests.

trade (Feenstra 1994; Broda and Weinstein 2006) to the welfare effects of tariffs (Romer 1994; Arkolakis et al. 2008) and sales taxes (Kroft et al. 2019) and the socially optimal level of variety (Spence 1976a; Spence 1976b; Dixit and Stiglitz 1977; Mankiw and Whinston 1986; and Dhingra and Morrow 2019).

We summarize the approach of Kroft et al. (2019) to estimating the effect of a small change in the number of varieties $J$ on consumer surplus. We show that with parallel demands, the change in consumer surplus can be measured by a "sufficient statistics" approach, which is based only on the aggregate demand. We first introduce some definitions following Kroft et al. (2019).

**Definition 3.** The consumer surplus is defined as the integral of aggregate demand

$$CS(p, J) = \int_p^\infty Q(s, J)ds. \tag{5}$$

The change in consumer surplus can be split into two effects. The "price effect" arises since market prices may change when firms enter or exit the market. The "variety effect" is by how much a new variety increases welfare since consumers exhibit a "love of variety", holding the effect on prices constant. In this application we focus on the "variety effect".

**Definition 4.** The "variety effect" is defined as

$$\Lambda(Q, J) \equiv \int_{P(Q,J)}^\infty \frac{\partial Q}{\partial J}(s, J)ds. \tag{6}$$

The variety effect depends on how aggregate demand responds to a change in variety. Up to a first-order approximation, the variety effect can be represented in terms of the causal effect of product variety on price or willingness-to-pay (see Lemma 1 in Kroft et al. (2019) for a formal statement and proof of this result).

$$\Lambda(Q, J) = \int_0^Q \frac{\partial P(t, J)}{\partial J}dt + O(dJ) \approx Q\overline{\frac{\partial P}{\partial J}}(Q, J)$$

where $\overline{\frac{\partial P}{\partial J}}(Q, J) = \frac{1}{Q}\int_0^Q \frac{\partial P}{\partial J}(t, J)dt$ is the average change in willingness to pay.

The above expression suggests that the exact measurement of the variety effect requires non-parametric identification of the inverse demand curve globally. However, this might be difficult to do in practice. This is where our theoretical results are useful. By assuming parallel demands, the expression of the variety effect can be simplified. In particular, Kroft et al. (2019) show:

$$\overline{\frac{\partial P}{\partial J}}(Q,J) = \frac{\partial P}{\partial J} = \frac{dP(Q,J)}{dJ} - \frac{\partial P(Q,J)}{\partial Q}\frac{dQ}{dJ} = \frac{dQ}{dJ}\left(\frac{\frac{dP(Q,J)}{dJ}}{\frac{dQ}{dJ}} - \frac{\partial P(Q,J)}{\partial Q}\right) \tag{7}$$

for all Q and J, where $\frac{\partial P(Q,J)}{\partial Q}$ denotes the slope of inverse demand when variety $J$ is held fixed, and $\frac{dP(Q,J)}{dJ}/\frac{dQ}{dJ}$ denotes the slope of inverse demand when $J$ is variable. The variety effect is then given by

$$\Lambda(J) = Q\frac{dQ}{dJ}\left(\frac{\frac{dP(Q,J)}{dJ}}{\frac{dQ}{dJ}} - \frac{\partial P(Q,J)}{\partial Q}\right) = Q\left[\frac{\frac{dp}{dz}}{\frac{dQ}{dz}} - \frac{\left.\frac{dp}{dz}\right|_J}{\left.\frac{dQ_J}{dz}\right|_J}\right]\frac{\frac{dQ}{dz}}{\frac{dJ}{dz}}$$

where $z$ is an instrumental variable observed in two scenarios: (1) when variety is held fixed and (2) when variety can vary. In a discrete choice model, once taste shocks are assumed to satisfy the conditions in Theorem 2 and thus the inverse demand is parallel, the welfare gain from changes in variety depends on three statistics: the demand elasticity with respect to variety, and the price elasticities of demand when $J$ is fixed and when $J$ is variable.

Moreover, equation (7) also provides the linear structure to identify the variety effect in an instrumental variable regression using a two-stage least squares (2SLS) approach detailed in Kroft et al. (2019). Observe:

$$\Delta P = \overline{\frac{\partial P}{\partial J}} * \Delta J + \frac{\partial P}{\partial Q}\Delta Q = \frac{\Lambda}{Q} * \Delta J + \frac{\partial P}{\partial Q}\Delta Q$$

which suggests the following linear regression model:

$$Q_i = \alpha + \left[\frac{\Lambda}{Q_0} * \frac{\partial Q}{\partial P}\right] J_i + \left[\frac{\partial Q}{\partial P}\right] P_i + \varepsilon_i$$

Since variety is endogenously determined, identifying the variety effect requires isolating exogenous variation in variety that is orthogonal to consumer preferences and other determinants of demand. Therefore, one can estimate the model using two different instruments, one for price and one for variety. For more details, see Kroft et al. (2019).

### 3.3 Numerical Simulations

Using the variety effect defined above, we assess the approximation in Theorem 2 by numerically simulating different random utility models and calculating the bias that arises from assuming demands are parallel. The simulation results are presented in Figure 1. Specifically, we simulate a model of a large number of consumers with utility over products given by equation (1). We choose $\alpha = 1$ and $y = 1$ in the simulation, and we consider four different shock distributions (Gumbel, Normal, Gamma, and Pareto). We then repeat this procedure for a range of different values of $J$ to assess how the bias from assuming parallel demand varies with $J$ when consider a hypothetical 20 percent increase in the number of products (from the initial value of $J$). We compute the welfare gains exactly using numerical methods and compare the exact welfare gain to the approximate gains implied by assuming parallel demands based on the formula in equation (6).

The results in Figure 1 show that the bias that arises from assuming parallel demands is a function of the number of varieties in the market, where bias is measured as the difference between the estimated (approximate) variety effect and the exact variety effect. The benchmark distribution is Gumbel where we know from theory that the demand curves are exactly parallel and therefore the bias is 0 for all $J$. For both the Normal and Exponential distributions, we find that the bias is small in magnitude and converges to 0 fairly quickly as the number of varieties increase. On the other hand, with a Pareto distribution, there is a bias of roughly 20 percent, which does not vanish as varieties increase. In this case, the variety effect computed using our sufficient statistics formula is a lower bound on the true variety effect.

## 4 Conclusion

The main contribution of this paper is to provide necessary and sufficient conditions for a discrete random utility model with an outside option to exhibit a "parallel demands" property. That is, when the random utility shocks are distributed i.i.d. and independent of the distribution of the shock of the outside option, the Gumbel distribution is the necessary and sufficient condition for the inverse demand curve to shift in parallel when variety changes. We use the Extreme Value Theorem to prove the necessity of this condition.

The class of models that follow exactly the Gumbel distribution is narrow, but our result is not a "knife-edge" result. To show the broader usefulness of the parallel demands property, we also provide an approximation result for the aggregate inverse demands to be asymptotically parallel. When the random utility shocks are i.i.d. and follow a distribution that is in the domain of attraction of the Gumbel distribution, the inverse demands can be approximated as parallel when the set of varieties is large. We support our theoretical results with a numerical simulation, and we also extend our results to a richer model that allows for correlated tastes within the inside market and differential substitutability across the inside and outside market.

Finally, we demonstrate an application of our theoretical results. By assuming parallel demands, consumers' change in surplus from changes in the number of varieties can be easily computed with three statistics: the demand elasticity with respect to variety, and the price elasticities of demand when $J$ is fixed and when $J$ is variable. The parallel demands property thus facilities identification and estimation of consumers' "love of variety", and Kroft et al. (2019) use this property to provide new empirical estimates of consumers' "love of variety" for products sold in grocery stores in the U.S.

We conclude by speculating that parallel demands might be a useful property to considser when studying other economic questions. For example, Spence (1975) shows how the profit-maximizing product quality compares to the socially optimal product quality. When adding the assumption of parallel demands – in terms of quality in this case, not variety – parallel demands may help provide new conditions for when the average willingness-to-pay for quality is equal to the marginal willingness-to-pay.

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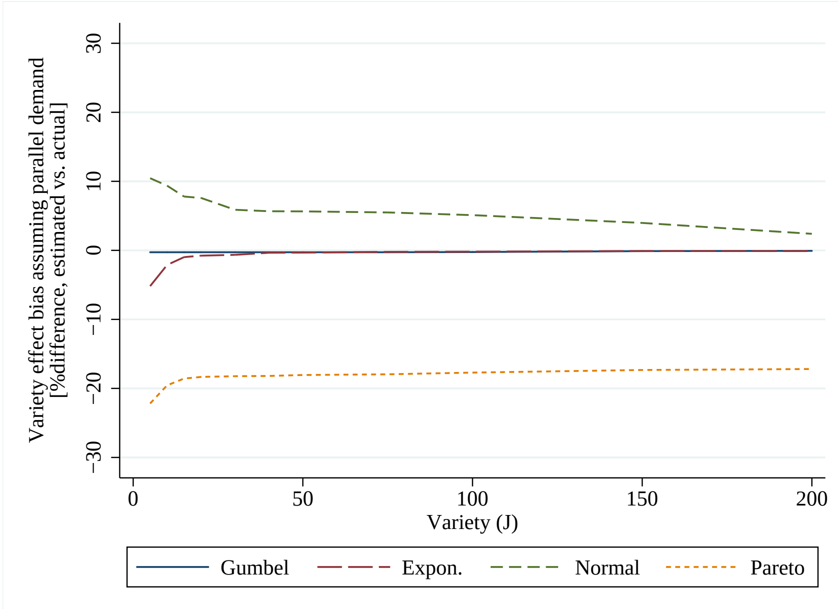
Figure 1: Approximate Parallel Demand Curves




Notes: This figure reports results from numerical simulations that are designed to evaluate the quality of the key approximation theorem (Theorem 2) in the main text. By simulating simple discrete choice models under different assumptions about distribution of the i.i.d. error terms and increasing number of varieties in the market, we calculate (exact) variety effect numerically and compare it to variety effect we would infer from assuming parallel demands. Consistent with result of Theorem 2, for distributions that satisfy assumptions of theorem, as J increases, the bias in variety effect from assuming parallel demands approaches zero.

# Appendix:

## Proofs of Claims, Propositions and Theorems

### Proof of Theorem 1

*Proof.* We have shown a sufficient condition to get inverse parallel demands is that the random utility shocks $(\varepsilon_{ij})$ are i.i.d. Gumbel, and independently of the distribution of $\varepsilon_{i0}$. If the shocks $(\varepsilon_{ij})$ are assumed to be i.i.d., then they have to be Gumbel in order to satisfy the inverse parallel demands condition as we now show.

Assuming the unique attribute is price and these are symmetric, the inverse demands when there are $J$ and $J+1$ varieties are parallel iff there exists $t$ such that for all $p$ then $Q(p, J) = Q(p+t, J+1)$, that is

$$\mathbb{P}(\varepsilon_{0m} < -p + \max_{1\leq j\leq J} \varepsilon_j) = \mathbb{P}(\varepsilon_{0m} < -p + t + \max_{1\leq j\leq J+1} \varepsilon_j).$$

Note that we normalize the common term $\alpha p - \delta$ of the inside market into $p$for simple notation.

Since $\varepsilon_{0m}$ is independent of $\max_{1\leq j\leq J} \varepsilon_j$ this can only be true if the distribution of the maxima is the same, that is

$$\max_{1\leq j\leq J} \varepsilon_j \quad \overset{d}{=} \quad t + \max_{1\leq j\leq J+1} \varepsilon_j$$

Let $F$ be the cdf of $\varepsilon$, then the equation above implies there exist $t(n)$ such that for all $x$:

$$F(x) = F^n(x + t(n)).$$

Iterating on both sides implies

$$F^{nm}(x + t(nm)) = F^{nm}(x + t(n) + t(m))$$

we recognize an instance of Hamel's functional equation $t(nm) = t(n) + t(m)$ which has solution $t(n) = c\log(n)$.[7] Therefore:

$$F(x) = F^y(x + c\log y),$$

letting $s = c\log y$, we get $F(0) = F^{e^{s/c}}(s)$, and so:

$$F(s) = e^{\log F(0) e^{-s/c}},$$

which is a Gumbel distribution with location parameter $c \log\log F(0)$ and dispersion parameter $c$. $\square$

## Proof of Theorem 2

*Proof.* Let the random utility shocks $(\sigma\varepsilon_j)$ be i.i.d. and distributed according to $F$ in the domain of attraction of the Gumbel distribution. Let $G(x) = exp[-exp(-x)]$ be the Gumbel distribution. Then there exist sequences $(a_n, b_n)$ such that

$$F^n(a_n x + b_n) \to G(x),$$

Furthermore, $\lim_{n\to\infty} \frac{a_n}{a_{[nt]}} = 1$ and $\lim_{n\to\infty} \frac{b_n - b_{[nt]}}{a_{[nt]}} = -c\log(t)$ for any $t > 0$ and some $c \in \mathbb{R}$ where $[nt]$ is the integer part of $nt$ (see Resnick (1987) Chapter 1). Since the convergence $F^n(a_n x + b_n) \to G(x)$ is uniform (see Resnick (1987) Chapter 0) and $F^n$ is uniformly continuous, then for any $\epsilon > 0$ there exists $\delta$ and $N(\delta, \epsilon)$ such that for all $x \in \mathbb{R}$ and all $J, K > N(\delta, \epsilon)$ we have $\left|\frac{a_K}{a_J} - 1\right| \leq \delta$ and

$$\begin{aligned}\left|F^J(a_J x + b_J) - F^K(a_J x + b_K)\right| &\leq \left|F^J(a_J x + b_J) - F^K(a_K x + b_K)\right| + \left|F^K(a_K x + b_K) - F^K(a_J x + b_K)\right| \\ &< \epsilon\end{aligned}$$

[7] It is easy to extend the formula for real numbers through rationals, note

$$F(x) = F^n(x + t(n)) = F^m(x + t(m))$$

implies

$$F(x) = F^{n/m}(x + t(n) - t(m)),$$

so we can consistently define $t(n/m) = t(n) - t(m)$.

Therefore, for any $p \in \mathbb{R}$

$$
\begin{aligned}
&|Q(p, J) - Q(p + b_K - b_J, K)| \\
&= \left| \mathbb{P}\left( \max_{j \in \{1,\ldots,J\}} u_{ij}(p) > u_{i0} \right) - \mathbb{P}\left( \max_{j \in \{1,\ldots,K\}} u_{ij}(p + b_K - b_J) > u_{i0} \right) \right| \\
&= \left| \int_{\mathbb{R}} \left( F^K\left(\eta_0 - \alpha(y-p) - \delta + b_K - b_J\right) - F^J\left(\eta_0 - \alpha(y-p) - \delta\right) \right) f_0(\eta_0) d\eta_0 \right| \\
&< \epsilon
\end{aligned}
$$

where $f_0$ is the probability density of $\eta_0 = u_{i0} - (1-\sigma)\nu_i$. We conclude that the inverse aggregate demands are asymptotically parallel. □

**Proof of Corollary 1**

*Proof.* We have shown a sufficient condition to get inverse parallel demands is that the random utility shocks $(\varepsilon_{ij})$ are i.i.d. Gumbel, independently of the size of $\sigma$, the distribution of $\nu_i$ and the distribution of $\varepsilon_{i0}$. If the shocks $(\varepsilon_{ij})$ are assumed to be i.i.d., then they have to be Gumbel in order to satisfy the inverse parallel demands condition as we now show.

Assuming the unique attribute is price and these are symmetric, the inverse demands when there are $J$ and $J+1$ varieties are parallel iff there exists $t$ such that for all $p$ then $Q(p, J) = Q(p+t, J+1)$, that is

$$\mathbb{P}(\varepsilon_{0m} < -p + \nu_m(1-\sigma_m) + (\sigma_m) \max_{1 \le j \le J} \varepsilon_j) = \mathbb{P}(\varepsilon_{0m} < -p + t + \nu_m(1-\sigma_m) + (\sigma_m) \max_{1 \le j \le J+1} \varepsilon_j).$$

Since $\varepsilon_{0m}$ and $\nu_m(1-\sigma_m)$ are independent of $\max_{1 \le j \le J} \varepsilon_j$ this can only be true if the distribution of the maxima is the same, that is

$$\max_{1 \le j \le J} \varepsilon_j \quad \overset{d}{=} \quad t + \max_{1 \le j \le J+1} \varepsilon_j$$

Let $F$ be the cdf of $\varepsilon$, then the equation above implies there exist $t(n)$ such that for all $x$:

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implies

$$F(x) = F^{n/m}(x + t(n) - t(m)),$$

so we can consistently define $t(n/m) = t(n) - t(m)$.