Parallel Inverse Aggregate Demand Curves in Discrete Choice Models

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Abstract

This paper highlights a previously-unnoticed property of commonly-used discrete choice models, which is that they feature parallel demand curves. Specifically, we show that in additive random utility models, inverse aggregate demand curves shift in parallel with respect to variety if and only if the random utility shocks follow the Gumbel (Type 1 Extreme Value) distribution. Using results from Extreme Value Theory, we provide conditions for other distributions to generate parallel demands asymptotically, as the number of varieties increases. We establish these results in the benchmark case of symmetric products, illustrate them using numerical simulations and show that they hold in extended versions of the model with correlated tastes and asymmetric products. Lastly, we provide a "proof of concept" of parallel demands as an economic tool by showing how to use parallel demands to identify the change in consumer surplus from an exogenous change in product variety.

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letting $x = 0$, $s = c \log y$, we get $F(0) = F^{e^{s/c}}(s)$, and so:

$$F(s) = e^{\log F(0) e^{-s/c}},$$

which is a Gumbel distribution with location parameter $c \log(-\log F(0))$ and dispersion parameter c. This derivation proves that the parallel demands condition implies that the random utility shocks $(\varepsilon_{ij})_{j=1}^{\infty}$ follow the Gumbel distribution. Moreover, if the random utility shocks $(\varepsilon_{ij})_{j=1}^{\infty}$ follow the Gumbel distribution then $F(x) = e^{\log F(0) e^{-x/c}}$ and $F^n(x) = e^{\log F(0) e^{log(n) - x/c}} = F\left(x - clog(n)\right)$ and so parallel demands hold:

$$\mathbb{P}\left(\varepsilon_{0m} < \delta - \alpha p + \max_{1 \le j \le J_0} \varepsilon_j\right) = \mathbb{P}\left(\varepsilon_{0m} < \delta - \alpha(p + clog(J_1) - clog(J_0)) + \max_{1 \le j \le J_1} \varepsilon_j\right).$$

□

Proof of Theorem 2

Proof. Let the random utility shocks (ε_j) be i.i.d. and distributed according to F in the domain of attraction of the Gumbel distribution. Let $G(x) = exp[-exp(-x)]$ be the Gumbel distribution. Then there exist sequences (a_n, b_n) such that

$$F^n(a_n x + b_n) \to G(x),$$

Furthermore, $\lim_{n \to \infty} \frac{a_n}{a_{[nt]}} = 1$ and $\lim_{n \to \infty} \frac{b_n - b_{[nt]}}{a_{[nt]}} = -c \log(t)$ for any $t > 0$ and some $c \in \mathbb{R}$ where $[nt]$ is the integer part of nt (see Resnick (1987) Chapter 1). Since the convergence $F^n(a_n x + b_n) \to G(x)$ is uniform (see Resnick (1987) Chapter 0) and F^n is uniformly continuous, then for any $\epsilon > 0$ there exists η and $N(\eta, \epsilon)$ such that for all $x \in \mathbb{R}$ and all $J_0, J_1 > N(\eta, \epsilon)$ we have $\left|\frac{a_{J_1}}{a_{J_0}} - 1\right| \le \eta$ and

$$\begin{aligned}\left|F^{J_0}(a_{J_0} x + b_{J_0}) - F^{J_1}(a_{J_0} x + b_{J_1})\right| &\le \left|F^{J_0}(a_{J_0} x + b_{J_0}) - F^{J_1}(a_{J_1} x + b_{J_1})\right| \\ &\quad + \left|F^{J_1}(a_{J_1} x + b_{J_1}) - F^{J_1}(a_{J_0} x + b_{J_1})\right| \\ &< \epsilon\end{aligned}$$

Therefore, for any $p \in \mathbb{R}$

$$
\begin{aligned}
&|Q(p, J_0) - Q(p + b_{J_1} - b_{J_0}, J_1)| \\
&= \left| \mathbb{P}\left(\max_{j \in \{1,\ldots,J_0\}} u_{ij}(p) > u_{i0} \right) - \mathbb{P}\left(\max_{j \in \{1,\ldots,J_1\}} u_{ij}(p + b_{J_1} - b_{J_0}) > u_{i0} \right) \right| \\
&= \left| \int_{\mathbb{R}} \left(F^{J_1}\left(\varepsilon_{i0} - \alpha(y-p) - \delta + \alpha(b_{J_1} - b_{J_0})\right) - F^{J_0}\left(\varepsilon_{i0} - \alpha(y-p) - \delta\right) \right) f_0(\varepsilon_{i0}) d\varepsilon_{i0} \right| \\
&< \epsilon
\end{aligned}
$$

where f_0 is the probability density of ε_{i0}. We conclude that the inverse aggregate demands are asymptotically parallel. □

Proof of Proposition 1

Proof. Redefine $\widetilde{\varepsilon}_{i0} = \varepsilon_{i0} - (1-\sigma)\nu_i$. Then the proof follows from Theorem 1. □

Proof of Theorem 3

Proof. Let F be the CDF of the random utility shocks. Define Condition A as: for all $(\delta_n)_{n=1}^{J_0}$ bounded vector of real non-negative numbers there exists $f((\delta_n)_{n=1}^{J_0})$ such that $F(x) = \Pi_{n=1} F\left(x - \delta_n + f((\delta_n)_{n=1}^{J_0})\right)$. Theorem 1 applies for vectors of constants $(\delta, \ldots, \delta)$ of any size, and shows that the only possible candidate CDF F that satisfies condition A must be Gumbel. Therefore if Condition A is going to hold for any $(\delta_n)_{n=1}^{J_0}$ bounded vector of real non-negative numbers, then F must be Gumbel. Condition A is a rephrasing of parallel WTP CDFs and so, Gumbel is necessary for parallel WTP CDFs.

Moreover, if $(\epsilon_{ij})_{j=1}^{\infty}$ are i.i.d. Gumbel then $\delta_j + \epsilon_{ij} \sim F_j(\mu_j, \beta)$ are also Gumbel, where μ_j is the position parameter of the Gumbel distribution and β is the scale parameter ($\{\mu_j\}$ is well defined, because $\{\delta_j\}$ is bounded.) Then

$$
\begin{aligned}
\mathbb{P}(\delta_j + \epsilon_{ij} < x) &= F_j(x) \\
&= exp\left(-exp\left(\frac{\mu_j - x}{\beta} \right) \right).
\end{aligned}
$$

Let $j^* = argmax_{j \in J_0}\{\delta_j + \epsilon_{ij}\}$, we have

$$\begin{aligned} F_{j^*}(x) &= \Pi_{j \in J_0} F_j(x) \\ &= exp(-\Sigma_{j \in J_0} exp(\frac{\mu_j - x}{\beta})) \\ &= exp(-exp(\frac{\mu - x}{\beta})), \end{aligned}$$

where $\mu = \beta log \sum_{j \in J_0} exp(\frac{\mu_j}{\beta})$.

Similarly, let $j^{**} = argmax_{j \in J_1}\{\delta_j + \epsilon_{ij}\}$ for $J_1 \neq J$. We have

$$F_{j^{**}}(x) = exp\left(-exp\left(\frac{\mu' - x}{\beta}\right)\right)$$

where $\mu' = \beta log \sum_{j \in J_1} exp(\frac{\mu_j}{\beta})$. The above derivation shows that we have parallel WTP distributions by letting

$$\begin{aligned} t_{J_1} &= \mu' - \mu \\ &= \beta log \frac{\sum_{j \in J_1} exp(\frac{\mu_j}{\beta})}{\Pi \sum_{j \in J_0} exp(\frac{\mu_j}{\beta})}. \end{aligned}$$

□

Proof of Theorem 4

Proof. Take (α_n, β_n) and the nondegenarate CDF H such that $\Pi_{j=1}^n F(\alpha_n x + \beta_n - \delta_j) \to H(x)$ for all x. Because

$$F^n(\alpha_n x + \beta_n) \leq \Pi_{j=1}^n F(\alpha_n x + \beta_n - \delta_j) \leq F^n(\alpha_n x + \beta_n)$$

and by continuity, there exists $\gamma_n \in [0, K]$ such that $\Pi_{j=1}^n F(\alpha_n x + \beta_n - \delta_j) = F^n(\alpha_n x + \gamma_n) \to H(x)$. But because F is in the domain of attratcion of the Gumbel, by Proposition 0.2 of Resnick (1987) there exists a and b such that $H(x) = G(ax + b)$ is a rescaling of the Gumbel distribution.

The rest of the proof follows exactly the same steps as the proof of Theorem 1, starting from $\Pi_{j=1}^n F(\alpha_n x + \beta_n - \delta_j) \to G(x)$. We have $\lim_{n\to\infty} \frac{a_n}{a_{[nt]}} = 1$ and $\lim_{n\to\infty} \frac{\gamma_n - \gamma_{[nt]}}{a_{[nt]}} = -c \log(t)$ for any $t > 0$ and some $c \in \mathbb{R}$ where $[nt]$ is the integer part of nt (see Resnick (1987) Chapter

1).

Since the convergence $\Pi_{j=1}^{n} F(\alpha_n x + \beta_n - \delta_j) = F^n(\alpha_n x + \gamma_n) \to G(x)$ is uniform (see Resnick (1987) Chapter 0) and F^n is uniformly continuous, then for any $\epsilon > 0$ there exists η and $N(\eta, \epsilon)$ such that for all $x \in \mathbb{R}$ and all $J_0, J_1 > N(\eta, \epsilon)$ we have $\left|\frac{a_{J_1}}{a_{J_0}} - 1\right| \leq \eta$ and

$$\begin{aligned}\left|F^{J_0}(a_{J_0}x + \gamma_{J_0}) - F^{J_1}(a_{J_0}x + \gamma_{J_1})\right| &\leq \left|F^{J_0}(a_{J_0}x + \gamma_{J_0}) - F^{J_1}(a_{J_1}x + \gamma_{J_1})\right| \\ &\quad + \left|F^{J_1}(a_{J_1}x + \gamma_{J_1}) - F^{J_1}(a_{J_0}x + \gamma_{J_1})\right| \\ &< \epsilon\end{aligned}$$

Therefore, for any $p \in \mathbb{R}$

$$\begin{aligned}&\left|\mathbb{P}\left(WTP_i(J_0) \leq x\right) - \mathbb{P}\left(WTP_i(J_1) \leq x + \frac{\gamma_{J_1} - \gamma_{J_0}}{\alpha}\right)\right| \\ &= \left|\mathbb{P}\left(\max_{j\in\{1,\ldots,J_0\}}\{\frac{\delta_j + \varepsilon_{ij} - \varepsilon_{i0}}{\alpha}\} \leq x\right) - \mathbb{P}\left(\max_{j\in\{1,\ldots,J_1\}}\{\frac{\delta_j + \varepsilon_{ij} - \varepsilon_{i0}}{\alpha}\} \leq x + \frac{\gamma_{J_1} - \gamma_{J_0}}{\alpha}\right)\right| \\ &= \left|\int_{\mathbb{R}} \left(F^{J_1}\left(\alpha x + \varepsilon_{i0} - \delta_j + \gamma_{J_1} - \gamma_{J_0}\right) - F^{J_0}\left(\alpha x + \varepsilon_{i0} - \delta_j\right)\right) f_0(\varepsilon_{i0}) d\varepsilon_{i0}\right| \\ &< \epsilon\end{aligned}$$

where f_0 is the probability density of ε_{i0}. We conclude that the willingness-to-pay densities are asymptotically parallel. □

Proof of Proposition 2

Proof. Assume parallel demands (Definition 1) and let $d(J_0, J_1)$ be such that $P(Q, J_0) + d(J_0, J_1) = P(Q, J_1)$. Then $\Lambda = \int_0^Q \Big(P(s, J_1) - P(s, J_0)\Big) ds = d(J_0, J_1) * Q$. □

Proof of Proposition 3

Proof. Observe:

$$\begin{aligned}d(J_0, J_1) &= p_1 - P(Q_1, J_0) \\ &= \left(\frac{p_1 - p_0}{Q_1 - Q_0} - \frac{P(Q_1, J_0) - p_0}{Q_1 - Q_0}\right)(Q_1 - Q_0)\end{aligned}$$

Now assume $(p(J), Q(J))_{J\in\mathbb{R}}$ is a continuously differentiable interpolation of $(p(J), Q(J))_{J\in\mathbb{N}}$

which exists by the Stone-Weierstrass theorem. Then by the Taylor approximation theorem:

$$\begin{aligned} d(J_0, J_1) &= \left(\frac{p_1 - p_0}{Q_1 - Q_0} - \frac{P(Q_1, J_0) - p_0}{Q_1 - Q_0} \right) (Q_1 - Q_0) \\ &= \left(\frac{\frac{dp}{dJ}}{\frac{dQ}{dJ}} - \frac{\partial P(Q, J)}{\partial Q} \right) \frac{dQ}{dJ} \triangle J + O((\triangle J)^2) \end{aligned}$$

□

Proof of Proposition 4

Proof. In the text. □

Proof of Proposition 5

Proof. Let $d = d(J_0, J_1)$. Observe by assumption $Q_{J_1}(\mathbf{p_{J_1}}) = Q_{J_0}(\mathbf{p_{J_0}} + (\rho - d)\mathbf{1_{J_0}})$, then the second part of the theorem follows directly from the first-order Taylor approximation:

$$Q_{J_1}(\mathbf{p_{J_1}}) = Q_{J_0}(\mathbf{p_{J_0}}) + (\rho - d)\frac{dQ_{J_0}(\mathbf{p_{J_0}} + t\mathbf{1_{J_0}})}{dt} + O\left((\rho - d)^2\right)$$

where $\frac{dQ_{J_0}(\mathbf{p_{J_0}} + t\mathbf{1_{J_0}})}{dt}$ is the directional derivative in the direction $\mathbf{1_{J_0}}$. And so

$$d = \left(\frac{\rho}{\Delta Q} - \left(\frac{dQ_{J_0}(\mathbf{p_{J_0}} + t\mathbf{1_{J_0}})}{dt} \right)^{-1} \right) \Delta Q + O\left((\rho - d)^2\right)$$

□